STABILITY OF PROCESSES WITH RESPECT TO TWO METRICS

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Let the functions $q_1(t), \ldots, q_k(t)$ determine the position of the points of a material system at an arbitrary instant of time $t \geqslant t_0$, where t_0 is a given initial instant of time. We denote the derivatives of these functions with respect to time by $q_{k+1}(t), \ldots, q_{2k}(t)$. Every set of functions $q_1(t), \ldots, q_{2k}(t)$ describes some motion of a material system. One of these motions, described by the set of functions $q_1^0(t), \ldots, q^0_{2k}(t)$, we shall call the undisturbed motion; all other possible motions of the material system under consideration we shall call disturbed motions.

Let the functions $Q_1(q_1, \ldots, q_{2k}, t), \ldots, Q_n(q_1), \ldots, q_{2k}, t)$ of the quantities q_s, and t be given. Assuming that the quantities q_s, Q_i are dimensionless, we introduce the notation

$$\rho_0(q(t), t) = \max_s |q_s(t) - q_s^0(t)| \qquad (1 \leqslant s \leqslant 2k)$$

$$\rho(q(t), t) = \max_i |x_i(t)| \qquad (1 \leqslant i \leqslant n)$$

$$x_i(t) = Q_i(q_1(t), \ldots, q_{2k}(t), t) - Q_i(q_1^0(t), \ldots, q_{2k}^0(t), t)$$

In this notation, Liapunov's [1, pp. 12-14] definition of stability, which was extended by Chetaev [2, pp. 9-11] to the case when the Q_i depend explicitly on t, is completely equivalent to the following definition: the undisturbed motion $q_s^0(t)$ is said to be stable with respect to the quantities Q_i if for every given $\epsilon > 0$ there exists a number $\delta > 0$ such that for every disturbed motion $q_s(t)$ which satisfies at the initial instant t_0 the condition

$$\rho_0(q(t_0), t_0) < \delta \qquad (0.1)$$

it is true that

$$\rho(q(t), t) < \varepsilon \qquad (0.2)$$

for all $t \geqslant t_0$.

The problems on stability are usually considered with the assumptions that the functions $x_i(t)$ are continuous for all $t \geqslant t_0$, while the functions Q_i are continuous in the q_s and t. Hereby the functions ρ_0 and ρ have the following properties:

0.1. The function $\rho_0(q(t), t)$ is a real nonnegative quantity which vanishes when $t \geqslant t_0$ for the undisturbed motion $q_s(t)$.

0.2. The function $\rho(q(t), t)$ is a real nonnegative quantity which vanishes when $t \geqslant t_0$ for the undisturbed motion $q_s(t)$.

0.3. Since the quantities Q_i are continuous in $q_1, \ldots, q_{2k}$, it follows that for every given $\epsilon > 0$ there exists a number $\delta > 0$ such that if $\rho_0(q(t_0), t_0) < \delta$ then $\rho(q(t_0), t_0) < \epsilon$.

0.4. Since the functions $x_i(t)$ are continuous along any disturbed motion $q_s(t)$, the function $\rho(q(t), t)$ is continuous for all $t \geqslant t_0$.

Having given the general definition of stability of the undisturbed motion $q_s^0(t)$ with respect to the quantities Q_i based on inequalities of the type (0.1) and (0.2), Liapunov developed his so-called direct method [1] on the basis of a definition of stability which follows from the general one under an auxiliary assumption [1, p. 17] on the properties of the functions Q_i. In our notation this assumption can be formulated in the following way:

0.5. For every given $\epsilon > 0$ there exists a number $\delta > 0$ such that if $\rho(q(t_0), t_0) < \delta$, then $\rho_0(q(t_0), t_0) < \epsilon$.

It is easily seen that if the five listed properties are satisfied then the above-given definition of stability is completely equivalent to the following definition: the undisturbed motion $q_s^0(t)$ is called stable if for every given $\epsilon > 0$ there exists a number $\delta > 0$ such that for every disturbed motion $q_s(t)$ which satisfies at the initial instant t_0 the condition

$$\rho(q(t_0), t_0) < \delta \qquad (0.3)$$

it is true that

$$\rho(q(t), t) < \varepsilon \qquad (0.4)$$

for all $t \geqslant t_0$.

This more restricted definition of stability, which is based on the inequalities (0.3) and (0.4), or on equivalent ones, has become the dominant one not only in the works of Liapunov but also in the works of later authors. In [3, p. 272], Liapunov calls attention to the existence of the more general definition based on inequalities of the type (0.1) and (0.2) for the particular case when $n < 2k$ and each of the quantities Q_i is simply one of the quantities q_s (a definition of stability with respect to some of the variables). In [4] theorems are formulated which are applicable to this definition of stability. These theorems deal with the properties of Liapunov's functions which are sufficient either for the stability or instability of the undisturbed motion with respect to some of the variables; two of these theorems (on stability and on asymptotic stability) are proved and illustrated by examples in [5]. In [6] a theorem is proved which is applicable to the general definition based on inequalities of the type (0.1) and (0.2). This theorem deals with properties of the integral of the equation of the disturbed motion which are sufficient for the stability of the undisturbed motion $q_s^0(t)$ with respect to the quantities Q_i. Appropriate examples to illustrate the theorems are also given in [6].

The present work is based on the omission of the property (0.5). We return to the general Liapunov definition of stability, generalizing it to the case when the processes which are being examined for stability are not necessarily connected by a finite number of variables q_s, Q_i; hereby the structures of the left-hand sides of the inequalities (0.1) and (0.2) may be arbitrary as long as the hypotheses 0.1 to 0.4 are satisfied (in problems on the mechanics of solid media, the left-hand sides of (0.1) and (0.2) may depend, for example, not only on the displacements and velocities of the points of the solid body, but also on the deformations, the stresses, temperatures and so on. [7, 8]). In order to avoid ambiguities with respect to the choice of the initial moments of time t_0, we consider it useful to indicate explicitly the set T_0 of possible initial moments t_0 in the definition of stability. Furthermore, in addition to the definition of stability, we shall consider also the definition of stability which is uniform on T_0. For each of the definitions there is proved a theorem of the direct method of Liapunov on the properties of functionals which are necessary and sufficient for the existence of a given type of stability or instability. Since it is not known in advance whether the processes under consideration possess the properties of continuity with respect to the initial values on a finite time interval, it is sensible to avoid wherever possible any assumptions on the boundedness or unboundedness of the time-interval T on which the stability is being investigated (see the note on the definition of stability on a finite time-interval in [9, p. 10]). The definition of stability here has many similarities to the definition of "correctness" in the theory of partial differential equations [10, pp. 28-32], [11, pp. 80.83]. In the proofs of the theorems on stability, the uniqueness of the solution is not assumed. On the contrary, certain results on the uniqueness are obtained as consequences of the existence of some functionals having certain definite properties.

1. Let Z be a set consisting of points z of an arbitrary nature. We say that a curve $z(z_0, t_0, t)$, beginning at z_0 at the time t_0, is given in Z if, to every value of the real parameter (time) t from the interval $t_0 \leqslant t < t_{00}$, or $t_0 < t < t_{00}$ (or, more briefly, (t_0, t_{00})), belonging to the time-interval T, there corresponds a definite point $z = z(z_0, t_0, t)$ from Z, and $z_0 = z(z_0, t_0, t_0)$. From the class of all possible curves, we select a subclass of curves which we shall call "processes". These curves must satisfy certain conditions and possess the following properties (in the numbering of the properties (axioms), definitions and theorems, the first number is the number of the section):

1.1. Every process, which is defined on (t_0, t_{00}), will still be a process if it is considered in an arbitrary interval (t_1, t_{11}) contained in (t_0, t_{00}).

1.2. If two processes have a point in common at the time t_1, then the composite curve, which consists of the points of one of the processes when $t \leqslant t_1$, and of the points of the other process when $t \geqslant t_1$, is also a process.

1.3. There exists a process which is defined on the entire interval T. We shall denote it by $z = z^0(t)$ and say that it is the undisturbed process; all other processes $z(z_0, t_0, t) \neq z^0(t)$ we shall call disturbed processes.

The process for which $t_0 = t_{00}$ will be called the degenerate process. If $t_{00} = \infty$, we shall say that the process is non-shortened.

By a point z one may mean an arbitrary set of parameters, functions and so on, which describe the state (mechanical, physical, chemical, etc.) of a material system at a given moment of time t, or one of its properties which may be of interest in some investigation. In such a case the process $z(z_0, t_0, t)$ becomes a mathematical abstraction of a mechanical, physical, chemical, etc. process which takes place within the material system in course of time.

2. If the point z at the time t belongs to some process, then we say that the pair (z, t) belongs to this process. Let us suppose that for every pair (z, t) of any given process there have been defined distances (metrics) $\rho_0(z, t)$ and $\rho(z, t)$ having the following properties:

2.1. The distance $\rho_0(z, t)$ is a real nonnegative number for every pair (z, t) of any process, and

$$\rho_0(z^0(t), t) \equiv 0, \qquad t \in T.$$

2.2. The distance $\rho(z, t)$ is a real nonegative number for every pair (z, t) of any process, and

$$\rho(z^0(t), t) = 0, \qquad t \in T.$$

2.3. The distance $\rho(z, t)$ is continuous with respect to the metric $\rho_0(z, t)$ [uniformly] on a given set $T_0 \leqslant T$, i.e. for every number $\epsilon > 0$ and for every $t_0 \leqslant T_0$, there exists a number $\delta(\epsilon, t_0) > 0$ [$\delta(\epsilon) > 0$] such that the inequality $\rho(z, t_0) < \epsilon$ will be satisfied for all pairs (z, t_0) belonging to any process and satisfying the condition $\rho_0(z, t_0) < \delta$.

2.4. The real function $\rho(z(z_0, t_0, t), t)$ of the argument t is continuous in t for every non-degenerate process $z(z_0, t_0, t)$ in the interval (t_0, t_{00}) where it is defined. Here $\rho(z(z_0, t_0, t), t)$ denotes the distance from the undisturbed process to the point z which belongs to the process $z(z_0, t_0, t)$ at the time t.

The distances ρ_0 and ρ satisfying the conditions 2.1, 2.2 and 2.3 can be obtained, for example, if one introduces into the set Z the metrics $\rho_1(z_1, z_2)$, $\rho_2(z_1, z_2)$ which satisfy the usual axioms of a metric space [12, p. 23] except for the requirement that $\rho_1(z_1, z_2) = 0$, $\rho_2(z_1, z_2) = 0$ imply $z_1 = z_2$, and if one assumes then that

$$\rho_0(z, t) = \rho_1(z, z^0(t)) + \rho_2(z, z^0(t)), \qquad \rho(z, t) = \rho_2(z, z^0(t))$$

The property 2.4 will also hold if the processes $z(z_0, t_0, t)$ are continuous with respect to the metric ρ_2. When we speak of the metrics ρ_0 and ρ in the sequel, we shall mean the distances $\rho_0(z, t)$ and $\rho(z, t)$ having the properties 2.1 to 2.4.

3. Comparing the undisturbed process $z^0(t)$ with the various processes $z(z_0, t_0, t)$ which begin at the moment t_0 belonging to the given set $T_0 \subseteq T$, we give the following definitions of stability:

Definition 3.1. The undisturbed process $z^0(t)$ is said to be stable with respect to the metrics ρ_0 and ρ on the interval T for a choice of the initial moment T_0 from the given set $T_0 \subseteq T$, if for every number $\epsilon > 0$ and for every initial moment $t_0 \in T_0$ there exists a number $\delta(\epsilon, t_0) > 0$ such that for every disturbed process $z(z_0, t_0, t)$ satisfying at the initial moment $t_0 \in T_0$ the condition

$$\rho_0(z_0, t_0) < \delta \tag{3.1}$$

it is true that

$$\rho(z(z_0, t_0, t), t) < \varepsilon \tag{3.2}$$

in the entire domain of definition of the process.

Definition 3.2. The undisturbed process $z^0(t)$ is said to be *uniformly stable*, with respect to the metrics ρ_0 and ρ on the interval T, on the given set $T_0 \subseteq T$ of initial moments t_0, if for every number $\epsilon > 0$, and for every initial moment $t_0 \in T_0$ there exists a number $\delta(\epsilon) > 0$, the same for all $t_0 \in T_0$, such that for every disturbed process $z(z_0, t_0, t)$, which satisfies condition (3.1) at the initial moment $t_0 \in T_0$ it is true that condition (3.2) is satisfied in the entire domain of definition of the process.

Conditions (3.1) and (3.2) imply the definitions of instability.

Definition 3.3. We shall say that the undisturbed process $z^0(t)$ is not stable with respect to the metrics ρ_0 and ρ on the interval T for a choice of the initial instant of time t_0 from the given set $t_0 \in T_0$, if for some number $\epsilon_1 > 0$ there exists at least one initial instant $T_0 \subseteq T$, such that for every number $\delta > 0$ there exists a disturbed process $z(z_0, t_0, t)$, $z_0 = z_0(\delta)$ for which condition (3.1) is satisfied at the indicated time t_0, and at some time $t_1 \geqslant t_0$ in the domain of definition of the process it is true that

$$\rho(z(z_0, t_0, t_1), t_1) \geqslant \varepsilon_1 \tag{3.3}$$

Definition 3.4. We shall say that the undisturbed process $z^0(t)$ is not uniformly stable with respect to the metrics ρ_0 and ρ on the interval T, on the set $T_0 \subseteq T$ of initial time-moments t_0, if for some number $\epsilon_1 > 0$ and for every $\delta > 0$, there exists a disturbed process $z(z_0, t_0, t)$ $(z_0 = z_0(\delta),\ \ t_0 = t_0(\delta) \in T_0)$, for which condition (3.1) holds at the initial moment t_0 and condition (3.3) is satisfied at some time $t_1 \geqslant t_0$ in the domain of definition of the process.

The mention of the metric ρ_0 in addition to the metric ρ (the analog of the quantity Q_i in Liapunov's general definition) in the definition of stability results from the circumstance that in various problems the measure of the initial disturbances can be related to the various characteristics of the processes under consideration. The metric ρ_0 serves as a measure of the initial disturbances. In Liapunov's general definition of stability the measure of the initial disturbances is not explicitly agreed upon, for it is assumed that this measure is once and for all connected in a definite manner with the deviations in the coordinates and velocities. In the case when $\rho \equiv \rho_0$ we shall say that we are dealing with stability with respect to the (single) metric ρ.

4. Let $R > 0$ be a fixed number. Any process $z(z_0, t_0, t)$, which is such that for every given t from the domain of definition (t_0, t_{00}), $\rho(z(z_0, t_0, t), t) < R$, is called an R-process. Let us denote by RZT

the set of pairs (z, t) for each of which there exists an R-process beginning at z at the moment t. We shall say that a real functional $f(z,t)$ is defined on RZT if to every pair $(z,t)\in RZT$ there corresponds a definite, unique real number $f(z, t)$. The value which is taken on by the functional $f(z, t)$ at the time t at the point z belonging to the R-process $z(z_0, t_0, t)$ will be denoted by $f(z(z_0, t_0, t)$. The metrics $\rho_0(z, t)$ and $\rho(z, t)$ are, obviously, examples of functionals.

Definition 4.1. The functional will be said to be positive-definite with respect to the metric ρ, if $f(z, t) \geqslant 0$ for every pair $(z,t)\in RZT$ and if for every positive number $\epsilon < R$ there exists a number $\mu(\epsilon) > 0$ (depending only on ϵ) such that the inequality $f(z, t) \geqslant \mu$ holds for every pair $(z,t)\in RZT$ satisfying the condition that $\rho(z, t) \geqslant \epsilon$ (if such pairs exist).

Definition 4.2. The functional $f(z, t)$ will be said to be nonincreasing if along any R-process $z(z_0, t_0, t)$ the function $f(z(z_0, t_0, t), t)$ of the time t does not increase with t in the domain of definition (t_0, t_{00}) of the R-process.

Definition 4.3. The functional $f(z, t)$ is said to be continuous with respect to the metric ρ_0 on the set T_0 if for every number $\epsilon > 0$ and for every moment of time $t_0\in T_0$ there exists a number $\delta(\epsilon, t_0) > 0$ such that the inequality $|f(z_0, t)| < \epsilon$ is true for every pair $(z_0,t_0)\in RZT$ satisfying the condition $\rho_0(z_0, t_0) < \delta$ (in consequence of the property 2.1, the functional $f(z, t)$ is continuous with respect to the metric ρ_0 on the set T_0, and satisfies the condition $f(z^0(t_0), t_0) = 0$ for every $t_0\in T_0$). If in this definition one can find one δ which will work for all $t_0\in T_0$, i.e. if δ depends only on ϵ $(\delta = \delta(\epsilon) > 0)$, then the continuity property 4.3 will be called *uniform* continuity on the set T_0.

Definition 4.4. The region $R(f > 0)$ is the set of pairs $(z,t)\in RZT$ for which $f(z, t) > 0$. The functional $f(z, t)$ will be said to be bounded in the region $R(f > 0)$ if there exists a positive number N such that for every pair (z, t) of this region it is true that $f(z, t) < N$.

Definition 4.5. We shall say that the functional $f(z, t)$ has a positive-definite derivative in the region $R(f > 0)$ if the derivative $df(z(z_0, t_0, t), t)/dt$ exists in the entire domain of definition (t_0,t_{00}) of every R-process $z(z_0, t_0, t)$ for every $f(z_0, t_0) > 0$, and if this derivative is bounded from below by some positive number (which will depend, in general, on the chosen R-process).

5. We shall prove the theorems of the direct method of Liapunov on stability and uniform stability, and also the propositions on asymptotic stability and on the connection between the properties of functionals and uniqueness.

Theorem 5.1. In order that the undisturbed process $z^0(t)$ may be stable with respect to the metrics ρ_0, ρ on the time-interval T for a given choice of the initial time-moment t_0 from the given set $T_0 \subseteq T$, it is necessary and sufficient that for some number $R > 0$ there exists on RZT a nonincreasing functional $f(z, t)$ which is positive-definite with respect to the metric ρ and continuous with respect to ρ_0 on the set T_0.

Theorem 5.2. Let the distance $\rho(z, t)$ be uniformly continuous with respect to the metric $\rho_0(z, t)$ on the set $T_0 \subseteq T$. In order that the undisturbed process $z^0(t)$ may be uniformly stable with respect to the metrics ρ_0 and ρ on the time-interval T in the set $T_0 \subseteq T$ of initial time moments t_0, it is necessary and sufficient that for some number $R > 0$ there exists on RZT a nonincreasing functional $f(z, t)$ which is positive-definite with respect to the metric ρ and uniformly continuous with respect to ρ_0 on the set T_0.

We shall carry out the proofs of Theorems 5.1 and 5.2 simultaneously by enclosing in square brackets those additions or modifications which pertain to uniform stability on T_0.

Necessity (in Theorems 5.1, 5.2, 6.1 and 6.2 the necessity is proved for the purpose of revealing the consistency of the requirements imposed on the functional $f(z, t)$ by the hypotheses of the theorems). Let R be a given positive number. We associate with each pair $(z, t) \in RZT$ the set $R(z, t)$ of all pairs (z_1, t_1) which belong to all possible R-processes that start at the point z at the time t.

We introduce the notation

$$f(z, t) = \sup \rho(z_1, t_1) \quad \text{for} \quad (z_1, t_1) \in R(z, t) \tag{5.1}$$

(A functional of the type (5.1) was used in [13]).

Because $R(z, t)$ is defined for every pair $(z_1, t_1) \in R(z, t)$ it follows that $(z_1, t_1) < R$. Hence, $f(z, t) \leqslant R$ and Formula (5.1) associates with each pair $(z, t) \in RZT$ a unique real number $f(z, t)$.

From the manner in which the set $R(z, t)$ was defined it follows that $(z, t) \in R(z, t)$, and from Formula (5.1) we deduce that $f(z, t) \geqslant \rho(z, t)$. From this and from property 2.2 it follows that the functional (5.1) is positive-definite with respect to the metric ρ (Definition 4.1), and that one can select for $\mu(\epsilon)$ the number ϵ, i.e. one may set $\mu(\epsilon) \equiv \epsilon$.

The functional (5.1) is nonincreasing (Definition 4.2). Indeed, let us consider some R-process $z(z_0, t_0, t)$ and any two of its points, $z' = z(z_0, t_0, t')$ and $z'' = z(z_0, t_0, t'')$ where $t'' \geqslant t' \geqslant t_0$. In view of properties 1.1 and 1.2, all pairs (z, t) of the set RZT, which belong to all possible processes beginning at the point z'' at the time t'', belong also to some processes beginning at the points z' at the time t'. Hence, $R(z', t') \supseteq R(z'', t'')$. From this and Formula (5.1) we obtain the result that $f(z', t') \geqslant f(z'', t'')$, or more precisely, that $f(z(z_0, t_0, t'), t') \geqslant f(z(z , t , t''), t'')$.

Thus, the functional (5.1) has the properties 4.1 and 4.2 independently of the fact whether the process $z^0(t)$ is stable or unstable. Let us assume that the process $z^0(t)$ is stable with respect to the metrics ρ_0 and ρ on the time-interval T for any choice of the initial time t_0 from the set $T_0 \subseteq T$ (uniformly stable on the set T_0). This means that for every $\epsilon_1 > 0$, and for every instant of time $t_0 \in T_0$ there exists a number $\delta_1(\epsilon_1, t_0) > 0$ [$\delta_1(\epsilon_1) > 0$] such that for every disturbed process $z(z_0, t_0, t)$ satisfying at the initial instant $t_0 \in T_0$ the condition $\rho_0(z_0, t_0) < \delta_1$ it is true that $\rho(z(z_0, t_0, t), t) < \epsilon_1$ in the entire domain of definition of the process. By the definition of the set $R(z, t)$ we now have the result that $\rho(z_1, t_1) < \epsilon_1$ for every pair $(z_1, t_1) \in R(z_0, t_0)$ and from Formula (5.1) it follows that $0 \leqslant f(z_0, t_0) \leqslant \epsilon_1$. Hence, the functional (5.1) is continuous with respect to the metric ρ_0 (uniformly continuous) on the set T_0. For the number δ we may choose, for example, the number $\delta(\epsilon, t_0) = \delta_1(\epsilon/2, t_0)$ [$\delta(\epsilon) = \delta_1(\epsilon/2)$].

Sufficiency. Let us assume that for some number $R > 0$ there exists on RZT a nonincreasing functional $f(z, t)$ which is positive-definite with respect to the measure ρ and continuous (uniformly continuous) with respect to the metric ρ_0 on T_0. Let a positive number ϵ be given. Obviously, in the proof of the theorem one may assume that $\epsilon < R$.

Case 1. For the given ϵ, there exist pairs $(z, t) \in RZT$, which satisfy the condition $\rho(z, t \geqslant \epsilon$.

The functional $f(z, t)$ is positive-definite with respect to the metric ρ (Definition 4.1). Hence, there exists a number $\mu(\epsilon) > 0$ such that $f(z, t) \geqslant \mu$ for every pair $(z, t) \in RZT$, satisfying the condition $\rho(z, t) \geqslant \epsilon$.

The functional $f(z, t)$ is continuous (uniformly continuous) with respect to ρ_0 on the set T_0 (Definition 4.3). Hence, for the number $\mu(\epsilon) > 0$ and for any instant of time $t_0 \in T_0$ there exists a number $\delta_1(\mu(\epsilon), t_0) > 0$ [$\delta\ (\mu(\epsilon)) > 0$] such that the inequality $|f(z_0, t_0)| < \mu$ is satisfied for every pair $(z_0, t_0) \in RZT$ satisfying the condition $\rho_0(z_0, t_0) < \delta_1$.

The distance $\rho(z, t)$ is continuous (uniformly continuous) with respect to the metric $\rho_0(z, t)$ on T_0 (property 2.3). Hence, for every given $\epsilon > 0$ and for every time-moment $t_0 \in T_0$ there exists a number $\delta_2(\epsilon, t_0) > 0$ [$\delta_2(\epsilon) > 0$] such that the inequality $\rho(z_0, t_0) < \epsilon$ is satisfied for every pair $(z_0, t_0) \in RZT$ satisfying the condition $\rho_0(z_0, t_0) < \delta_2$.

Let us introduce the notation

$$\delta(\varepsilon, t_0) = \min(\delta_1(\mu(\varepsilon), t_0), \delta_2(\varepsilon, t_0)), \qquad [\delta(\varepsilon) = \min(\delta_1(\mu(\varepsilon)), \delta_2(\varepsilon))]$$

and let us show that for every initial instant of time $t_0 \in T_0$ every disturbed process $z(z_0, t_0, t)$ satisfying condition (3.1) in the entire domain of definition will satisfy condition (3.2). Suppose this is not true, and that for some instant of time $t_0 \in T_0$ there exists a disturbed process $z(z_0, t_0, t)$ for which condition (3.1) holds, while at some instant $t_1 \geqslant t_0$ from the domain of definition of the given process we have

$$\rho(z(z_0, t_0, t_1), t_1) \geqslant \varepsilon \tag{5.2}$$

In view of the choice of the number $\delta \leqslant \delta_1$, it follows from condition (3.1) that

$$f(z(z_0, t_0, t_0), t_0) = f(z_0, t_0) < \mu \tag{5.3}$$

In view of the choice of the number $\delta \leqslant \delta_2$, it follows from (3.1) also that

$$\rho(z(z_0, t_0, t_0), t_0) = \rho(z_0, t_0) < \varepsilon$$

From this and condition (5.2) it can be seen that $t_0 < t_1$, i.e. the considered process is non-degenerate. Let us consider the set of all those values t from the interval $t_0 \leqslant t \leqslant t_1$ for which it is true that $\rho(z(z_0, t_0, t), t) \geqslant \epsilon$. This set is non-empty, for by property (5.2) it contains the value $t = t_1$. Let us denote the lower boundary of this bounded non-empty set by t_{00} and the corresponding point of the process under consideration by $z_{00} = z(z_0, t_0, t_{00})$. From the continuity of the function $\rho(z(z_0, t_0, t), t)$ (property 2.4), and from the definition of the lower boundary, we find that for every t of the interval $t_0 \leqslant t \leqslant t_{00}$ it is true that $\rho(z(z_0, t_0, t), t) < \epsilon$, and that

$$\rho(z(z_0, t_0, t_{00}), t_{00}) = \rho(z_{00}, t_{00}) = \varepsilon$$

Therefore, (see the beginning of the proof)

$$f(z(z_0, t_0, t_{00}), t_{00}) = f(z_{00}, t_{00}) \geqslant \mu \tag{5.4}$$

Since $\epsilon < R$, the considered disturbed process $z(z_0, t_0, t)$ is an R-process in the interval $(t_0 \leqslant t \leqslant t_{00})$. For this R-process the inequality (5.3) holds when $t = t_0$; when $t = t_{00} > t_0$ we have the inequality (5.4) which contradicts the condition that the functional $f(z, t)$ is nonincreasing (Definition 4.2).

Case 2. For the given ϵ there exists no pair $(z, t) \in RZT$ which satisfies the condition $\rho(z, t) \geqslant \epsilon$.

The distance $\rho(z, t)$ is continuous (uniformly continuous) with respect to the metric $\rho_0(z, t)$ on the set T_0 (property 2.3). Therefore, for the given number $\epsilon > 0$ and for every instant of time $t_0 \in T_0$ there exists a number $\delta(\epsilon, t_0) > 0$ $[\delta(\epsilon) > 0]$ such that the inequality $\rho(z_0, t_0) < \epsilon$ holds for every pair $(z_0, t_0) \in RZT$ satisfying the condition that $\rho_0(z_0, t_0) < \delta$. For the found δ, condition (3.2) is satisfied for every disturbed process $z(z_0, t_0, t)$ satisfying condition (3.1) in the entire domain of definition of the process. Let us assume that this is not true, and that there exists a disturbed process $z(z_0, t_0, t)$ for which condition (3.1) holds, and that due to the choice of δ

$$\rho(z(z_0, t_0, t_0), t_0) = \rho(z_0, t_0) < \varepsilon \tag{5.5}$$

but that for some instant $t_1 \geqslant t_0$ from the domain of definition of the process, condition (5.2) holds. Comparing (5.5) with (5.2), we see that $t_1 > t_0$, i.e. the process under consideration is non-degenerate. Because of the continuity of the function $\rho(z(z_0, t_0, t), t)$ (property 2.4) there exists an instant of time t_* in $t_0 \leqslant t \leqslant t_1$ such that the point $z_* = z(z_0, t_0, t_*)$ satisfies the equation $\rho(z_*, t_*) = \epsilon$. Since $\epsilon < R$ and $(z_*, t_*) \in RZT$, we have arrived at a contradiction to the initial hypothesis of Case 2. Theorems 5.1 and 5.2 have thus been proved.

Supplement on the asymptotic properties of the functional $f(z, t)$. Let us assume that the undisturbed process $z^0(t)$ is stable with respect to the metrics ρ_0 and ρ on the time-interval T and uniformly stable on the set $T_0 = T$ of the initial instants t_0, and let there exist unabridged processes $z(z_0, t_0, t)$ which approach asymptotically (in the metric ρ_0) the undisturbed process $z^0(t)$, i.e. they satisfy the condition

$$\rho_0(z(z_0, t_0, t), t) \to 0 \text{ when } t \to \infty \tag{5.6}$$

Then, for some number $R > 0$, there exists on RZT a nonincreasing functional $f(z, t)$ which is positive-definite with respect to ρ and uniformly continuous, with respect to the metric ρ_0, on the set $T_0 = T$ (this was already proved in Theorem 5.2) which vanishes along any R-process that satisfies condition (5.6); this means that

$$f(z(z_0, t_0, t), t) \to 0 \text{ when } t \to \infty \tag{5.7}$$

(This result will still have to be proved.)

Proof: Let $\epsilon > 0$ be given, and let $z(z_0, t_0, t)$ be any R-process satisfying condition (5.6). We must show that there exists an instant of time $t_1 \geqslant t_0$ such that for all $t \geqslant t_1$ $|f(z(z_0, t_0, t), t)| < \epsilon$ along all the R-processes under consideration.

The functional $f(z, t)$ is uniformly continuous with respect to the metric ρ_0 on the set $T_0 = T$ (Definition 4.3). Hence, for the given $\epsilon > 0$, and for every instant of time $t \in T$ there exists a number $\delta(\epsilon) > 0$, depending only on ϵ, such that $|f(z, t)| < \epsilon$ for every pair $(z, t) \in RZT$ satisfying the condition $\rho_0(z, t) < \delta$. In view of condition (5.6), and by means of the found number $\delta(\epsilon)$, one can find an instant of time $t_1 \geqslant t_0$ such that $\rho_0(z_1, t_1) < \delta$ at the point $z_1 = z(z_0, t_0, t_1)$, and hence, $|f(z_1, t_1)| < \epsilon$. Since the functional $f(z, t)$ is positive-definite and nonincreasing (Definitions 4.1, 4.2) for all instants of time $t \geqslant t_1$, the inequality

$$0 \leqslant f(z(z_0, t_0, t), t) \leqslant f(z_1, t_1) < \varepsilon$$

holds along any one of the R-processes under consideration, which was to be proved.

Supplement on the asymptotic behavior of perturbed processes. Suppose that for some number $R > 0$ there exists on RZT a functional $f(z, t)$ which is positive-definite with respect to the metric ρ, and suppose there exist unabridged R-processes $z(z_0, t_0, t)$ along which the functional $f(z, t)$ vanishes, i.e. satisfies condition (5.7). Then every such R-process approaches asymptotically with respect to the metric ρ a certain undisturbed process $z^0(t)$, i.e. the following condition holds:

$$\rho(z(z_0, t_0, t), t) \to 0 \text{ when } t \to \infty \tag{5.8}$$

Proof. Let $z(z_0, t_0, t)$ be some unabridged R-process satisfying condition (5.7) and let ϵ be a given positive number such that $\epsilon < R$. We must show that there exists an instant of time $t_1 \geqslant t_0$ such that $\rho(z(z_0, t_0, t), t) < \epsilon$ for all $t \geqslant t_1$ on the R-processes under consideration.

If the considered R-process does not contain any pairs (z, t) which satisfy the condition $\rho(z, t) \geqslant \epsilon$, then the proof is trivial and $t_1 = t_0$.

Suppose that the considered R-process contains a pair (z, t) for which $\rho(z, t) \geqslant \epsilon$. The functional $f(z, t)$ is positive-definite with respect to the metric ρ (Definition 4.1). Therefore, for every given positive number ϵ there exists a number $\mu(\epsilon) > 0$, depending on ϵ, such that $f(z, t) \geqslant \mu$ for every pair $(z, t) \in RZT$, satisfying the condition $\rho(z, t) \geqslant \epsilon$. The functional $f(z, t)$ vanishes along the considered R-process $z(z_0, t_0, t)$. Hence, for $\mu > 0$ there exists an instant of $t_1 \geqslant t_0$ such that $f(z(z_0, t_0, t), t) < \mu$ for all $t \geqslant t_1$. But then for all $t \geqslant t_1$ we must have

$\rho(z(z_0, t_0, t), t) < \epsilon$, for in the opposite case we would have

$$\varepsilon \leqslant \rho(z(z_0, t_0, t), t) < R$$

for some $t \geqslant t_1$, and, hence, we would have $f(z(z_0, t_0, t), t) \geqslant \mu$. This establishes the relation (5.8).

Supplement on the connection between the properties of functionals and uniqueness. An undisturbed process $z^0(t)$ will be said to possess on the set T_0 the property of ρ-uniqueness to the right if, for every process $z(z^0(t), t_0, t)$ starting at the point $z^0(t_0)$ of the undisturbed process at the time $t_0 \in T_0$ it is true that $\rho(z(z^0(t_0), t_0, t), t) \equiv 0$ in the entire domain of definition (it is obvious that because of the properties 1.1 to 1.3 the undisturbed process $z^0(t)$ which has the property of ρ-uniqueness to the right on T_0 will also possess this property on every subset of the interval T which lies to the right of the lower boundary of T_0).

Consequence of the theorems on stability. If there exists a functional $f(z, t)$ which has the properties specified in the hypotheses on Theorems 5.1 and 5.2, then the undisturbed process $z^0(t)$ has on the set T_0 the property of ρ-uniqueness to the right. Indeed, suppose that this is not so and that there exists a process $z(z^0(t_0), t_0, t)$, $t_0 \in T_0$, for which at some time $t_1 > t_0$, the following condition holds:

$$\rho(z(z^0(t_0), t_0, t_1), t_1) = \varepsilon_1 > 0 \tag{5.9}$$

The disturbed process under consideration satisfies for every given $\delta > 0$ condition (3.1) since we have, by property 2.1, that

$$\rho_0(z(z^0(t_0), t_0, t_0), t_0) = \rho_0(z^0(t_0), t_0) = 0 \quad (< \delta)$$

But by (5.9), with $\epsilon = \epsilon_1$, condition (3.2) is violated at the time $t_1 > t_0$. Hence, under the stated hypothesis, the undisturbed process $z^0(t)$ cannot be stable with respect to the metrics ρ_0 and ρ on the interval T for a choice of the initial moments from T_0. This contradicts the conclusion of the theorems of stability (sufficiency).

It is obvious that the theorem of [6] is a consequence of the theorem 5.1 (sufficiency) for, by the hypothesis made in [6], the integral (1.5) of [6] satisfies the conditions of Theorem 5.1 if one introduces the metrics $\rho_0 = \Sigma x_i^2$, $\rho = \Sigma y_i^2$ (the positive-definiteness of ϕ with respect to the metric ρ follows from the inequality (1.6) of [6] and from the hypothesis on the positive-definiteness of the function Φ; the nonincreasing nature of ϕ follows from the fact that ϕ is an integral; the continuity of ϕ with respect to the metric ρ_0 follows from the hypothesis on the continuity of the function ϕ with respect to all its variables).

6. Let the following conditions be satisfied:

6.1. For some number $R_1 > 0$ we have on $R_1 ZT$ the condition of uniqueness to the right. This means that for every pair $(z_0, t_0) \in R_1 ZT$ any two R_1-processes $z_1(z_0, t_0, t)$ and $z_2(z_0, t_0, t)$, which begin at one point z_0 at the same time t_0, coincide for $t \geqslant t_0$ in the common part of their domains of definition (this hypothesis is used only in the proof of necessity).

6.2. For every pair $(z, t) \in R_1 ZT$ there exists an unabridged disturbed process which begins at the point z at the time t (this hypothesis is used only in the proof of sufficiency).

Under the above conditions the following theorems on instability are valid. These theorems generalize the theorem of Chetaev [2] to the case of two metrics.

Theorem 6.1. In order that the undisturbed process $z^0(t)$ may not be stable with respect to the metrics ρ_0 and ρ on the time-interval T for any choice of the initial instant of time t_0 from the given set $T_0 \subset T$ it is necessary and sufficient that for some $R > 0$, there exist on RZT a bounded functional $f(z, t)$ which has a positive-definite derivative in the region $R(f > 0)$, and that for at least one instant of time $t_0 \in T_0$ and for every given $\delta > 0$ there exist a pair (z_0, t_0) in the region $R(f > 0)$ satisfying the condition $\rho_0(z_0, t_0) < \delta$ (the point $z_0 = z_0(\delta)$ depends, in general, on the chosen δ; the instant of time t_0 is the same for all δ).

Necessity. Let us take any positive number $R \leqslant R_1$. Let (z_0, t_0) be a definite pair of the set RZT. From all possible R-processes $z(z_0, t_0, t)$ which begin at the point z_0 at the time t_0, we select the one for which the domain of definition (t_0, t_{00}) is the largest. We call this process the maximal process. In view of the assumed uniqueness to the right (condition 6.1), there corresponds a unique definite maximal R-process $z(z_0, t_0, t)$ to each given pair $(z_0, t_0) \in RZT$. We shall say that the maximal R-process $z(z_0, t_0, t)$ can be extended to the boundary $\rho(z,t)=R$ if its domain of definition is of the form $t_0 \leqslant t \leqslant t_{00}$, $t_{00} < \infty$, and if there exists a disturbed process $z(z_0, t_0, t)$ for which

$$\rho(z(z_0, t_0, t_{00}), t_{00}) = R$$

Along any maximal R-process $z(z_0, t_0, t)$, in the entire domain, of its definition (t_0, t_{00}), we set

$$f(z(z_0, t_0, t), t) = e^{t-t_{00}}, \quad \text{if } z(z_0, t_0, t) \text{ can be extended to the boundary } (z, t) = R$$
$$f(z(z_0, t_0, t), t) \equiv 0, \quad \text{if } z(z_0, t_0, t) \text{ cannot be extended to the boundary } (z, t) = R \tag{6.1}$$

(A functional of the type (6.1) was used in [14]). In this manner there is associated with each pair $(z, t) \in RZT$ a unique real number $f(z, t)$ (the uniqueness is a consequence of the uniqueness to the right, while the existence of the number is obvious from Formulas (6.1)).

Formulas (6.1) show that the condition $0 \leqslant f(z, t) < 1$ is satisfied for every pair $(z, t) \in RZT$; hence, the functional (6.1) is bounded in the region $R(f > 0)$ (Definition 4.4).

Along any R-process $z(z_0, t_0, t)$ for which $f(z_0, t_0) > 0$, the functional (6.1) takes on values which are given by the first of Formulas (6.1). The derivative

$$\frac{d}{dt} f(z(z_0, t_0, t), t) = \exp(t - t_{00})$$

exists everywhere and is bounded from below by the number $\exp(t_0 - t_{00}) > 0$. Hence, the functional (6.1) has a positive-definite derivative in the region $R(f > 0)$ (Definition 4.5).

Thus, independently of the stability or instability of the undisturbed process $z^0(t)$, the functional (6.1) satisfies the requirements of Definitions 4.4 and 4.5 (we call attention to the fact that this does not imply that the region $R(f > 0)$ is not empty for the functional (6.1)).

Suppose that the undisturbed process $z^0(t)$ is not stable (Definition 3.3). Then for every number $\epsilon_1 > 0$ there exists at least one initial instant of time $t_0 \in T_0$, such that for every $\delta > 0$ there exists a disturbed process $z(z_0, t_0, t)$, $z_0 = z_0(\delta)$ for which condition (3.1) holds at the indicated initial instant t_0 and for which condition (3.3) is valid at some instant of time $t_1 \geqslant t_0$ from the domain of definition of the process.

Let a positive number δ be given. It is obvious that in the proof of the theorem one can consider δ to be so small that, in view of the continuity of the distance $\rho(z, t)$ with respect to the metric $\rho_0(z, t)$ (property 2.3), the inequality (3.1) must imply the inequality

$$\rho(z_0, t_0) < R \tag{6.2}$$

where R is the smaller of the two positive numbers ϵ_1 and R_1. For the given δ we take a disturbed process $z(z_0, t_0, t)$ which satisfies conditions (3.1) and (3.3). Then, because of the sufficient smallness of δ, condition (6.2) will also be satisfied, and because of the choice of the number $R \leqslant \epsilon_1$, we will have the inequality

$$\rho(z(z_0, t_0, t_1), t_1) \geqslant R \tag{6.3}$$

Comparing (6.2) and (6.3) we see that $t_1 > t_0$ and that the considered process $z(z_0, t_0, t)$ is non-degenerate. Because of the continuity of the function $\rho(z(z_0, t_0, t), t)$ (property 2.4), we can conclude, on the basis of (6.2) and (6.3), that there exists a finite instant of time $t_{00} > t_0$ such that the disturbed process under consideration satisfies the conditions

$$\rho(z(z_0, t_0, t), t) < R \quad \text{when} \quad t_0 \leqslant t < t_{00}, \qquad \rho(z(z_0, t_0, t_{00}), t_{00}) = R$$

Hence, for the pair (z_0, t_0), this disturbed process $z(z_0, t_0, t)$ on the interval $t_0 \leqslant t \leqslant t_{00}$ is a maximal R-process which can be extended to the boundary $\rho(z, t) = R$. The functional (6.1) constructed on RZT takes on positive values along this R-process. In particular, $f(z_0, t_0) = \exp(t_0 - t_{00}) > 0$; the pair (z_0, t_0) belongs to the region $R(f > 0)$ and satisfies, for any given number $\delta > 0$, the condition $\rho_0(z_0, t_0) < \delta$, which was to be proved.

Sufficiency. Let $R > 0$ be given, and suppose that there exists on RZT a bounded functional $f(z, t)$ which possesses a positive-definite derivative in the region $R(f > 0)$. Furthermore, let us suppose that for some instant of time $t_0 \in T_0$ and for every given number $\delta > 0$ there exists a pair (z_0, t_0) from the region $R(f > 0)$ which satisfies the condition $\rho_0(z_0, t_0) < \delta$ (the point $z_0 = z_0(\delta)$ depends, in general, on the chosen δ; the instant t_0 is the same for all δ). Let us assume that the undisturbed process $z^0(t)$ is stable with respect to the metrics ρ_0 and ρ on the time-interval T for a choice of the initial instant of time t_0 from the given set $T_0 \subset T$ (Definition 3.1). Then, for the number $R > 0$, and for the above-indicated instant of time $t_0 \in T_0$, there exists a number $\delta(R, t_0) > 0$ such that for every disturbed process $z(z_0, t_0, t)$, satisfying at the initial instant t_0 condition (3.1), it is true that $\rho(z(z_0, t_0, t), t) < R$ in the entire domain of definition. This means that the process is an R-process.

On the basis of the condition of the theorem, one can find, for the found δ, a pair (z_0, t_0) from the region $R(f > 0)$ which satisfies the condition $\rho_0(z_0, t_0) < \delta$; and, because of condition 6.2, one can find an unabridged process $z(z_0, t_0, t)$ which corresponds to this pair and which is an R-process, as was already shown. Since the functional $f(z, t)$

possesses in the region $R(f > 0)$ a positive-definite derivative (Definition 4.5), it follows that for some number $\nu > 0$ the next displayed inequalities hold along the considered unabridged R-process $z(z_0, t_0, t)$ for all t in the interval $t_0 \leqslant t < \infty$:

$$\frac{d}{dt} f(z(z_0, t_0, t), t) \geqslant \nu$$

$$f(z(z_0, t_0, t), t) = f(z_0, t_0) + \int_{t_0}^{t} dt \frac{d}{dt} f(z(z_0, t_0, t), t) \geqslant f(z_0, t_0) + \nu(t - t_0)$$

Since $f(z_0, t_0) > 0$ and $\nu > 0$, the right-hand side of the last inequality is positive for every t from the interval $t_0 \leqslant t < \infty$ and, for large enough t, it exceeds any previously given number. This contradicts the hypothesis on the boundedness of the functional $f(z, t)$ in the region $R(f > 0)$ (Definition 4.4). This completes the proof of Theorem (6.1).

Theorem 6.2. In order that the undisturbed process $z^0(t)$ may not be stable with respect to the metrics ρ_0, ρ on the time-interval T and not be uniformly stable on the given set $T_0 \subseteq T$ of initial moments of time t_0, it is necessary and sufficient that for some number $R > 0$ there exist on RZT a functional $f(z, t)$ which is bounded and has a positive-definite derivative in the region $R(f > 0)$, and that for every number $\delta > 0$ there exist a pair (z_0, t_0), of the region $R(f > 0)$, which satisfies the condition $\rho_0(z_0, t_0) < \delta$, $t_0 \in T_0$ (the point $z_0 = z_0(\delta)$ and the instant of time $t_0 = t_0(\delta)$ depend, in general, on the chosen δ).

Necessity. In the proof of Theorem 6.1 it was established that if condition 6.1 was satisfied then, independently of the stability or instability of the undisturbed process $z^0(t)$, the functional (6.1) will be bounded and possess a positive-definite derivative in the region $R(f > 0)$ (Definitions 4.4 and 4.5).

Let us assume that the undisturbed process $z^0(t)$ is not uniformly stable on the set T_0 (Definition 3.4). Then, for some number $\epsilon_1 > 0$, and for every number $\delta > 0$ there exists a disturbed process

$$z(z_0, t_0, t) \ (t_0 = t_0(\delta) \in T_0, \ z_0 = z_0(\delta)),$$

which satisfies conditions (3.1) and (3.3). Making use of this, and repeating word for word the end (from just before Formula (6.2)) of the proof of necessity of Theorem 6.1, we establish that the functional (6.1), constructed on RZT where R is the smaller of the two positive numbers ϵ_1 and R_1, also possesses the third property of Theorem 6.2.

Sufficiency. Let R be some positive number. Let us suppose that there exists on RZT a functional $f(z, t)$ which is bounded and possesses a positive-definite derivative in the region $R(f > 0)$. Furthermore, let us suppose that for every $\delta > 0$ there exists a pair (z_0, t_0) of the region $R(f > 0)$ which satisfies the condition $\rho_0(z_0, t_0) < \delta$, $t_0 \in T_0$ (the point $z_0 = z_0(\delta)$, and the instant of time $t_0 = t_0(\delta)$ depend, in general, on the chosen δ). Let us assume that the undisturbed process $z^0(t)$ is stable with respect to the metrics ρ_0 and ρ on the time-interval T and uniformly stable on the given set $T_0 \subseteq T$ of initial instants of time t_0 (Definition 3.2). Then, for $R > 0$, and for every initial instant $t_0 \in T_0$ there exists a number $\delta(R) > 0$ (the same for all instants $t_0 \in T_0$) such that every disturbed process $z(z_0, t_0, t)$ which at the initial instant of time $t_0 \in T_0$ satisfies condition (3.1) will satisfy also the condition $\rho(z(z_0, t_0, t), t) < R$ in the entire domain of definition; namely, it will be an R-process. Making use of this and repeating word for word the end (from the second paragraph on the sufficiency) of the proof of Theorem 6.1, we establish the existence of an unabridged R-process along which there is violated the condition on the boundedness of the functional $f(z, t)$ in the region $R(f > 0)$. This proves Theorem 6.2.

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